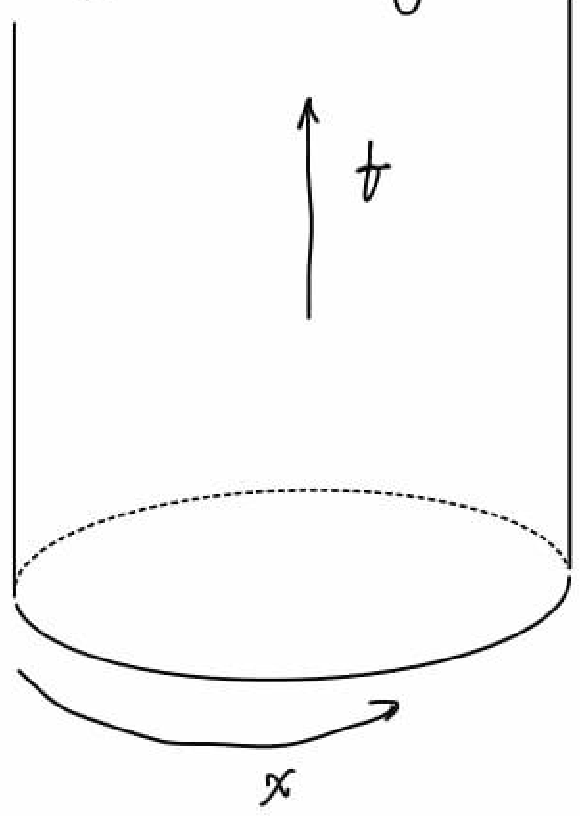
Quench problem: The parameter of the Hamiltonian or Lagrangian fastly changes so that the adiabatic process breaks down (fast relative to other time scales in the system. In the so-called Floquet CFT, we use an abrupt quench, which means H_0 switches to H_1 suddenly, and then switch H_2 after some time. so on and so forth. How to design H_0 and H_1 is a problem of CFT.

Suppose our system lives on a cylinder. Then the Hamiltonian is written as:



$$H_0 = -\int_0^L T(x) + \bar{T}(x)\, dx$$

We would like a sequence of non-commutable Hamiltonian, which we denote as $H_0, H_1, H_2 \cdots H_j \cdots$. So we deform the Hamiltonian as:

$$H_j = -\int_0^L f_j(x) \left(T(x) + \bar{T}(x) \right) dx$$

$f_j(x)$ represents a sequence of real function. Then how to design $f_j(x)$? We would like our H_j can be decomposed to some conserved quantity L_n, since we know the commutation relation $[L_m, L_n]$ in CFT. So that we can give the result of sth like $U^\dagger(T) H_0 U(T)$ through this commutation relation. So the next step is to expand $T(x)$ as L_n, and then design $f_j(x)$ accordingly to arrive at our aim. Be careful that we are working on the cylinder:

$$\left.\begin{aligned} z_p &= e^{-iz\cdot\frac{2\pi}{L}} \\ T_p &= \sum_n \frac{L_n}{z_p^{n+2}} \\ T(z) &= \left(\frac{2\pi}{L}\right)^2 \left[-z_p^2 T_p(z_p) + \frac{c}{24} \right] \end{aligned}\right\} \quad T(z) = \left[-\sum_n L_n \cdot e^{i\frac{2\pi n}{L} z} + \frac{c}{24} \right] \left(\frac{2\pi}{L}\right)^2$$

Then we have: $H_j = \int_0^L f_j(x) \left[\sum_n L_n e^{i\frac{2\pi n}{L} z} + \sum_n \bar{L}_n e^{i\frac{2\pi n}{L}\bar{z}} - \frac{c}{12} \right] dx \cdot \left(\frac{2\pi}{L}\right)^2$

Obviously we would like to choose $f_j(x)$ to be like $e^{i\frac{2\pi q}{L}x}$, so that we can integral out L_n or $\bar{L}_n$. The simplest setup would be:

$$f_j(x) = \underbrace{C_{0j}}_{\text{real number}} + C_{1j} e^{i\frac{2\pi q}{L}x} + C_{1j}^* e^{-i\frac{2\pi q}{L}x}$$

So that in every period, we switch the coffecient C_{0j} and C_{ij} while keeping q as constant. Notice that we have to maintain $f_j(x)$ as a real function so that H_j is real. Insert this $f_j(x)$ into integral:

$$\int_0^L e^{i\frac{2\pi q}{L}x} \cdot e^{i\frac{2\pi n}{L}x}\, dx = L \cdot \delta_{n+q}$$

$$\Downarrow$$

$$H_j = \frac{4\pi^2}{L}\left[C_{0j}\left(L_0+\bar{L}_0-\frac{c}{12}\right) + C_{ij}\left(L_{-q}+\bar{L}_{-q}\right) + C_{ij}^*\left(L_q+\bar{L}_q\right)\right]$$

We can seperate H_j into chiral part and anti-chiral part:

$$H_{j,\text{chiral}} = \frac{4\pi^2}{L}\left[C_{0j}\left(L_0-\frac{c}{12}\right) + (C_{ij}+C_{ij}^*)\frac{L_q+L_{-q}}{2} + i(C_{ij}-C_{ij}^*)\frac{L_q-L_{-q}}{2i}\right]$$

$$= \frac{4\pi^2}{L}\left[v_j^0\left(L_0-\frac{c}{12}\right) + v_j^+ L_{q,+} + v_j^- L_{q,-}\right]$$

↳ v_j^0, v_j^+, v_j^- are all real numbers, as one can easily see

And similar for $H_{j,\text{anti-chiral}}$. We write in this way since $\frac{L_q+L_{-q}}{2}$ and $\frac{L_q-L_{-q}}{2i}$ are both Hermitian operator ($L_{-q}=L_q^+$). Notice that $[L_n, \bar{L}_m]=0$. So that $e^{-iH_jt_j}$ can be decomposed to $e^{-iH_{j,c}t_j}\, e^{-iH_{j,ac}t_j}$. Let's see how this expression simplifies our calculation. For ex:

$$e^{iH_jt}H_0e^{-iH_jt} \xRightarrow[\text{equivalent to calculating}]{} e^{iH_{j,c}t}L_0e^{-iH_{j,c}t} = L_0 + it[H_{j,c}, L_0] + \frac{(it)^2}{2!}[H_{j,c}, [H_{j,c}, L_0]] + \cdots$$

$$[H_{j,c}, L_0] = \frac{4\pi^2}{L}\left(v_j^+[L_{q,+}, L_0] + v_j^-[L_{q,-}, L_0]\right) = \frac{4\pi^2}{L}\left(v_j^+L_{q-} + v_j^-L_{q+}\right)\cdot iq$$

$$[L_{q+}, L_0] = \frac{1}{2}[L_q+L_{-q}, L_0] = \frac{1}{2}(qL_q - qL_{-q}) = i\frac{q}{2i}(L_q-L_{-q}) = iqL_{q-}$$

$$[L_{q-}, L_0] = \frac{1}{2i}[L_q-L_{-q}, L_0] = \frac{q}{2i}(L_q+L_{-q}) = -iqL_{q+}$$

$$\Longrightarrow [H_{j,c}, [H_{j,c}, L_0]] = \cdots$$

In principle one can give the result in this way, but it's difficulty to do the calculation. Here we just justify the choice of $f_j(x)$.

In general we should find $e^{iH_jt_j}O(z,\bar{z})e^{-iH_jt_j}$. Notice that $e^{-iH_jt_j}$ is nothing

↳ Primary operator

but a unitary transformation, so that:

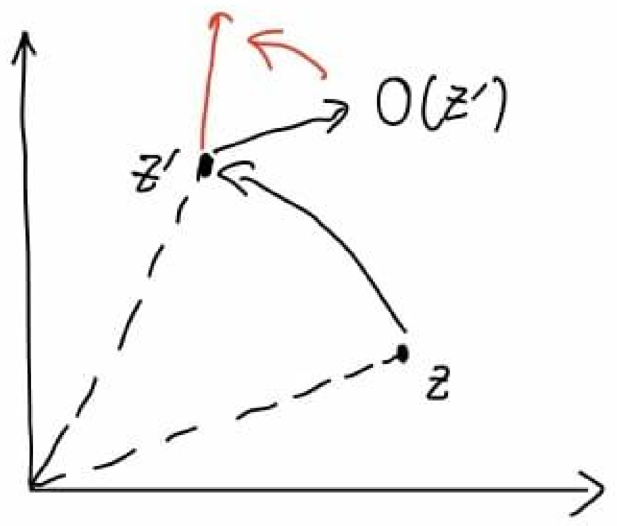
$$e^{iH_jt_j}O(z,\bar z)e^{-iH_jt_j} = \left(\frac{dz'}{dz}\right)^h\left(\frac{d\bar z'}{d\bar z}\right)^{\bar h}O(z',\bar z')$$

No prime here

One can compare this with coordinate transformation:

$$\left(\frac{dz'}{dz}\right)^{-h}\left(\frac{d\bar z'}{d\bar z}\right)^{-\bar h}O(z,\bar z) = O'(z',\bar z')$$

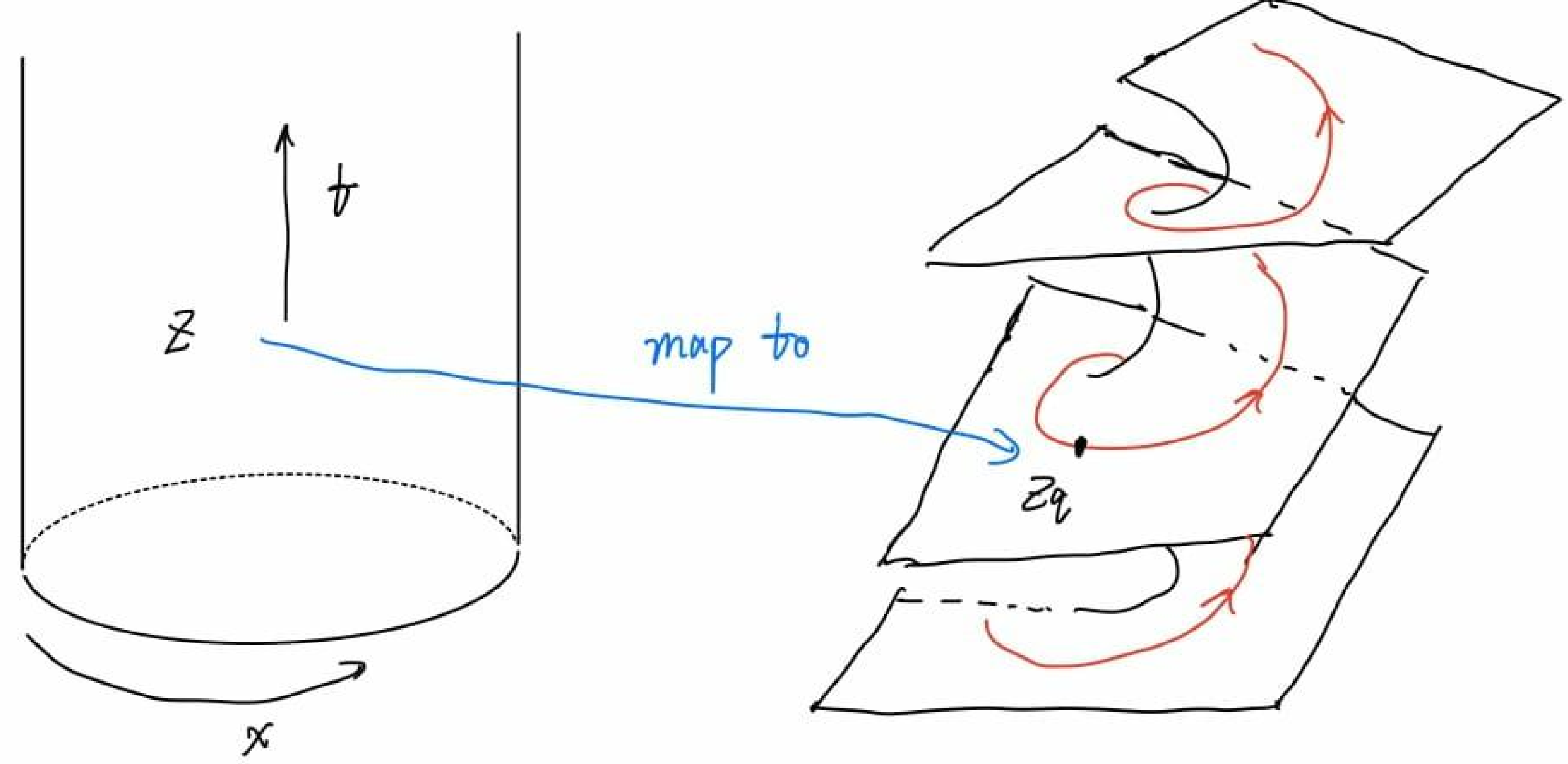
前者是主动观点，后者是被动观点．以旋转为例：



么正变换首先将场点从 z 移到 z'，然后再对 $O(z')$ 作主动旋转．从坐标变换来说，这相当于把坐标系反向旋转，因此要取逆变换 $\left(\frac{dz'}{dz}\right)^h$ 而不是 $\left(\frac{dz'}{dz}\right)^{-h}$．这 Justify 了上述等式．自然，从 OPE 的角度也能证明该式，见于附录．这实际上相当于 $U^\dagger V_i U = R_{ij}V_j$ 的场论版本，只是 $O(z,\bar z)$ 是一个局域算符

Now the problem is to find the corresponding coordinate transformation of $e^{-iH_jt_j}$. To do this, we need a somewhat indirect method: First, we map a cylinder to the q-sheet Riemann surface by:

$$z_q = e^{-i\frac{2\pi q}{L}z} = e^{(t-ix)\frac{2\pi q}{L}}$$



Similar to the case of $z \longrightarrow z_p$, the radial direct corresponds to time, and the tangent direct corresponds to space. We need the q-sheet to explain $z_q = e^{-i\frac{2\pi q}{L}z}$ because we want every point on a spatial slice of cylinder maps to different points on the q-sheet. Under this transformation:

$$\begin{cases} T(z) = \left(\frac{2\pi q}{L}\right)^2\left[-z_q^2\, T_q(z_q) + \frac{c}{24}\right], \quad dz_q = -i\frac{2\pi q}{L}\, e^{-i\frac{2\pi q}{L}z}\, dz \Longrightarrow dz = \frac{dz_q}{z_q}\cdot i\frac{L}{2\pi q} \\ L_n = -\frac{L}{4\pi^2}\int_0^L T(x)\cdot e^{-i\frac{2\pi n}{L}x}\, dx + \frac{c}{24}\delta_{n,0} \end{cases}$$

Be careful that $\int_0^L dx = -\oint \frac{i\, dz_q}{z_q}\cdot\frac{L}{2\pi q}$

$$L_0 = -\frac{L}{4\pi^2}\int_0^L T(x)\, dx + \frac{c}{24} = +\frac{L}{4\pi^2}\oint \left(\frac{2\pi q}{L}\right)^2\left[-z_q^2 T_q + \frac{c}{24}\right]\frac{dz_q}{z_q}\cdot\frac{L}{2\pi q}\, i + \frac{c}{24}$$

$$= +\frac{L}{4\pi^2}\cdot\frac{2\pi q}{L}\, i\oint\left(-z_q T_q + \frac{c/24}{z_q}\right)dz_q + \frac{c}{24} = \frac{q}{2\pi i}\oint T_q\, z_q\, dz_q$$

The integral is translational invariant, so that $e^{-i\frac{2\pi q}{L}x}$ can be promoted to $e^{-i\frac{2\pi q}{L}z} = z_q$

$$L_q = -\frac{L}{4\pi^2}\int_0^L T(x)\, e^{-i\frac{2\pi q}{L}x}\, dx = \frac{q}{2\pi i}\oint T_q\, z_q\cdot z_q\, dz_q = \frac{q}{2\pi i}\oint T_q\cdot z_q^2\, dz_q$$

$$L_{-q} = -\frac{L}{4\pi^2}\int_0^L T(x)\, e^{-i\frac{2\pi q}{L}x}\, dx = \frac{q}{2\pi i}\oint T_q\, dz_q$$

So that:

$$H_{j,\text{chiral}} = \frac{4\pi^2}{L}\left[v_j^0 L_0 + v_j^+ L_{q,+} + v_j^- L_{q,-}\right] - \frac{4\pi^2}{L}\cdot\frac{v_j^0 c}{12}$$

$$= \frac{4\pi^2}{L}\cdot\frac{q}{2\pi i}\oint\left(v_j^0 z_q + v_j^+\frac{z_q^2+1}{2} + v_j^-\frac{z_q^2-1}{2i}\right)T_q - \text{const}$$

$$= \frac{2\pi q}{Li}\oint\left[\left(\frac{v_j^+}{2} + \frac{v_j^-}{2i}\right)z_q^2 + v_j^0 z_q + \left(\frac{v_j^+}{2} - \frac{v_j^-}{2i}\right)\right]T_q - \text{const}$$

Since $H_{j,\text{chiral}}$ is composed by conserved quantity L_0, L_{q+}, L_{q-}, itself is also a conserved quantity. Any conserved quantity can be written as $J(z) = \oint T(z)\xi(z)$ where $\xi(z)$ is the generator of the transformation induced by $J(z)$. Then the infinitesimal version of this tranformation is $z' = z + \xi(z)\cdot d\theta$. In our case here, we know that:

$$\frac{dz_q'}{d\theta} = \xi(z_q) = a z_q^2 + b z_q + a^*$$

It's relatively difficult to solve z' as a function of z from this expression. Here we just check the result: such kind of $\xi(z)$ corresponds to a Möbius trans:

$$z' = \frac{\alpha z + \beta}{\beta^* z + \alpha^*} \quad \text{with } |\alpha|^2 - |\beta|^2 = 1$$

Let's see if it is the case. First we parameterize α and β as:

$$\alpha = e^{i\varphi} ch(\theta) \ , \quad \beta = e^{i(\varphi+\Delta\varphi)} sh(\theta)$$

We have two parameters characterizing the transformation. First we take $\theta \longrightarrow 0$, and collect terms to linear order of θ:

$$z' = \frac{e^{i\varphi} z + e^{i(\varphi+\Delta\varphi)}\theta}{e^{-i(\varphi+\Delta\varphi)}\theta z + e^{-i\varphi}} = e^{i2\varphi} \frac{z + e^{i\Delta\varphi}\theta}{1 + e^{-i\Delta\varphi}\theta z} = e^{i2\varphi}\left(z + e^{i\Delta\varphi}\theta\right)\left(1 - e^{-i\Delta\varphi}\theta z + O(\theta^2)\right)$$

$$= e^{i2\varphi}\left[z + \theta\left(e^{i\Delta\varphi} - e^{-i\Delta\varphi} z^2\right)\right]$$

Then we take $\varphi \longrightarrow 0$: $\quad z' = \left(1 + i\cdot 2\varphi + O(\varphi^2)\right)\left[z + \theta\left(e^{i\Delta\varphi} - e^{-i\Delta\varphi} z^2\right)\right]$

$$= z + (2iz)\cdot\varphi + \left(e^{i\Delta\varphi} - e^{-i\Delta\varphi} z^2\right)\theta + O(\varphi\theta)$$

Since φ and θ are now both infintesimally small, we can assume $\varphi = \lambda\theta$, where $\lambda \sim O(1)$ is a parameter of the vector $\xi(z)$, just as $\Delta\varphi$. z' is then:

$$z' = z + i\left(ie^{-i\Delta\varphi} z^2 + 2\lambda z - ie^{i\Delta\varphi}\right)\theta = z + i\left(\eta z^2 + 2\lambda z + \eta^*\right)\theta$$

Compare it with: $\quad \xi(z_q) = \left(\frac{v_j^+}{2} + \frac{v_j^-}{2i}\right) z_q^2 + v_j^0 z_q + \left(\frac{v_j^+}{2} - \frac{v_j^-}{2i}\right) = a z_q^2 + bz + a^*$

As one can see, the generator of $z' = \frac{\alpha z + \beta}{\beta^* z + \alpha^*}$ basically has the same form, despite some constant and normalization. We can tune the form of $\xi(z_q)$ to match it with Möbius trans:

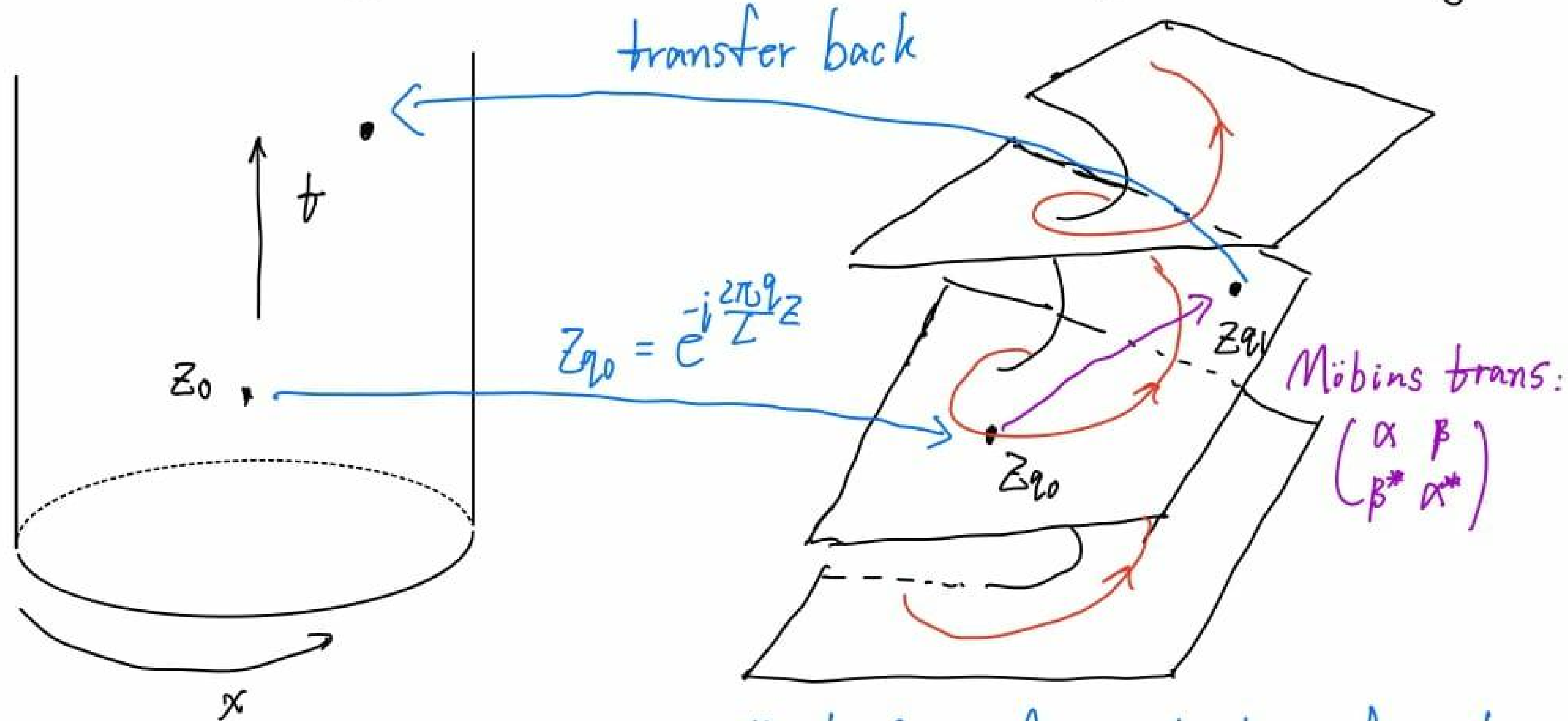
$$\xi(z_q) = \frac{1}{2}\left[\left(v_j^+ - iv_j^-\right) z_q^2 + 2v_j^0 z_q + \left(v_j^+ + iv_j^-\right)\right]$$

$$= \frac{\sqrt{v_j^{+2} + v_j^{-2}}}{2}\left[e^{-i\phi} z_q^2 + \frac{2v_j^0}{\sqrt{v_j^{+2} + v_j^{-2}}} z_q + e^{i\phi}\right], \quad \text{where } \phi = \arctan\frac{v_j^-}{v_j^+}$$

Then we have $\begin{cases} \eta = e^{-i\phi} \implies \Delta\varphi - \frac{\pi}{2} = \arctan\frac{v_j^-}{v_j^+} \\ \lambda = \frac{v_j^0}{\sqrt{v_j^{+2} + v_j^{-2}}} \end{cases}$

Other constant coffecient can be absorbed into θ. Then one can find the relationship between α, β and v_j^0, $v_j^\pm$. The expression can be referered to P66. (175)

We emphasize here that the Möbius trans should be operated on the q-sheet Riemann surface, correspondingly the operator $O(z,\bar{z})$ should also be the version defined on q-sheet surface. This does not matter that much: we can always transfer back to the cylinder to find the real physics quantity.



This is just an illustration of point transformation
Operator trans should follow the formula in P8

Then a sequence of time evolution $\prod_j e^{-iH_j t_j}$ can be transferred to a sequence of Möbius transformation. We know that Möbius trans forms a group, which means:

$$\begin{cases} z' = \dfrac{a_1 z + b_1}{c_1 z + d_1} \\ z'' = \dfrac{a_2 z' + b_2}{c_2 z' + d_2} \end{cases} \Longrightarrow z'' = \frac{a_3 z + b_3}{c_3 z + d_3}, \text{ where } \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

As one can check by direct plugging in. This simplifies calculation a lot: by $z_0 \to z_1 \to z_2 \cdots \to z_n$, we already know that $z_0 \to z_n$ is still a Möbius trans. and the coffecient of trans can be given by group theory.

Let's formulate our idea:

$U(T) = U_n(t_N)\, U_{n-1}(t_{N-1}) \cdots U_2(t_2)\, U_1(t_1)$, where $U_i(t_i) = e^{-iH_i t_i}$

Each $U_i(t_i)$ represents a Möbius trans. denoted as $M_i = \begin{pmatrix} \alpha_i & \beta_i \\ \beta_i^* & \alpha_i^* \end{pmatrix}$

More explicitly: $U^\dagger(t_i)\, O_q(z_q, \bar{z}_q)\, U(t_i) = \left(\dfrac{dz_q'}{dz_q}\right)^h \left(\dfrac{d\bar{z}_q'}{d\bar{z}_q}\right)^{\bar{h}} O(z_q', \bar{z}_q')$. with $z_q' = \dfrac{\alpha_i z_q + \beta_i}{\beta_i^* z_q + \alpha_i^*}$

Notice that: $U_{i-1}^{+} U_i^{+} O_q(z_q, \bar{z}_q) U_i U_{i-1} = \left(\frac{dz_q'}{dz_q}\right)^h \left(\frac{d\bar{z}_q'}{d\bar{z}_q}\right)^{\bar{h}} U_{i-1}^{+} O_q(z_q', \bar{z}_q') U_{i-1}$

$$= \left(\frac{dz_q'}{dz_q}\right)^h \cdot \left(\frac{dz_q''}{dz_q'}\right)^h \cdots\cdots = \left(\frac{dz_q''}{dz_q}\right)^h \left(\frac{d\bar{z}_q''}{d\bar{z}_q}\right)^{\bar{h}} O(z_q'', \bar{z}_q'')$$

where $z_q \longrightarrow z_q' : M_i$, $z_q' \longrightarrow z_q'' : M_{i-1}$, so that $z_q \longrightarrow z_q'' : M_{i-1} M_i$

which means $U_i U_{i-1}$ corresponds to $M_{i-1} M_i$ (注意指标顺序颠倒了)

So that: $U(T) = U_n U_{n-1} \cdots U_2 U_1 \Longleftrightarrow \Pi = M_1 M_2 \cdots M_{n-1} M_n = \begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix}$

with $|A|^2 - |B|^2 = 1$

$U^{+}(T) O_q(z_q, \bar{z}_q) U(T) = \left(\frac{dz_q'}{dz_q}\right)^h \left(\frac{d\bar{z}_q'}{d\bar{z}_q}\right)^{\bar{h}} O(z_q', \bar{z}_q')$, $z_q' = \frac{Az_q + B}{B^* z_q + A}$

Despite primary field O_q , we would also be interested in the evolution of the energy density $T(z_q)$. But notice that $T(z)$ is not a primary field:

$$U^{+}(T) T(z_q) U(T) = \left(\frac{dz_q'}{dz_q}\right)^2 T(z_q') + \frac{c}{12} Sch(z_q', z_q)$$

In general we should add a Schwarzian derivative as a consequence of coordinate transformation. Fortunately, $Sch(z_q', z_q) = 0$ when $z_q \longrightarrow z_q'$ is a Möbius trans, so we can get rid of this complex term (While we also lose the chance of exploring thermalization of Casimir effect). Keep in mind that what we need to derive is $U^{+}(T) T(z) U(T)$, which means we need to transfer back to cylinder:

$T(z) = \left(\frac{2\pi q}{L}\right)^2 \left[-z_q^2 T(z_q) + \frac{c}{24}\right]$.

Transfer of $O(z_q)$ back to $O(z)$ is basically same with here

$$U^{+} T(z) U = \left(\frac{2\pi q}{L}\right)^2 \left[-z_q^2 U^{+} T(z_q) U + \frac{c}{24}\right] = \frac{\pi^2 q^2 c}{6L^2} - \left(\frac{2\pi q}{L}\right)^2 z_q^2 \cdot \left(\frac{dz_q'}{dz_q}\right)^2 T(z_q')$$

$$z_q' = \frac{Az_q + B}{B^* z_q + A^*} , \quad \frac{dz_q'}{dz_q} = \frac{A(B^* z_q + A^*) - (Az_q + B)B^*}{(B^* z_q + A^*)^2} = \frac{|A|^2 - |B|^2}{(B^* z_q + A^*)^2} = \frac{1}{(B^* z_q + A^*)^2}$$

Now we need to take the average value $\langle U^{+} T(z) U \rangle$, so the problem comes to which state we should take. Still we take the easiest example : ground state. The problem is that what is $\langle T(z_q')\rangle$? Note that it's not zero :

$$T(z) = \left(\frac{2\pi q}{L}\right)^2\left[-z_q^2 T(z_q) + \frac{c}{24}\right] = \left(\frac{2\pi}{L}\right)^2\left[\underbrace{-z_p^2 T(z_p)}_{\text{map of cylinder to plane}} + \frac{c}{24}\right]$$

(Or 1-sheet Riemann surface)

$$\Rightarrow q^2\left[-z_q^2 T(z_q) + \frac{c}{24}\right] = -z_p^2 T(z_p) + \frac{c}{24}$$

$$\Rightarrow q^2\left[-z_q^2\langle T(z_q)\rangle + \frac{c}{24}\right] = \frac{c}{24} \Rightarrow \langle T(z_q)\rangle = \frac{c}{24}\frac{q^2-1}{q^2}\frac{1}{z_q^2}$$

Recall that we define $\langle T(z_p)\rangle = 0$ for ground state. So that we can give the Casimir energy on the cylinder. We have then:

$$\langle T(z_q')\rangle = \frac{c}{24}\frac{q^2-1}{q^2}\frac{1}{z_q'^2} = \frac{c}{24}\frac{q^2-1}{q^2}\frac{(B^* z_q + A^*)^2}{(Az_q+B)^2}$$

$$\langle U^\dagger T(z) U\rangle = \frac{\pi^2 q^2 c}{6L^2} - \frac{\pi^2 c}{6L^2}(q^2-1)\frac{z_q^2}{(Az_q+B)^2(B^* z_q + A^*)^2}$$

(注意这里z_q不带量纲，所以量纲是对的)

where $z_q = e^{-i\frac{2\pi q}{L}z}$ is the evolution start point. For simplicity we set $t=0$ at the start point. So that $z_q = e^{-i\frac{2\pi q}{L}x}$, and $\frac{1}{z_q} = z_q^*$. Then:

$$\frac{z_q^2}{(Az_q+B)^2(B^* z_q + A^*)^2} = \frac{1}{(Az_q+B)^2(A^* z_q^* + B^*)^2} = \frac{1}{|Az_q+B|^4} = \frac{1}{|Ae^{-i\frac{2\pi q}{L}x}+B|^4}$$

$$\boxed{-\langle U^\dagger T(z) U\rangle = \frac{\pi^2 c}{6L^2}\left[\frac{q^2-1}{|Ae^{-i\frac{2\pi q}{L}x}+B|^4} - q^2\right]}$$

Which represents the energy density evolution on the cylinder (minus sign is due to the definition $H_0 = -\int_0^L T + \bar{T}\, dx$). If we integrate it over the spatial slice $t=0$, we can give the whole energy of system at $t=T$:

$$\boxed{E(T) = -\int_0^L \langle U^\dagger T(z) U\rangle + \langle U^\dagger \bar{T}(\bar{z}) U\rangle\, dx = \frac{\pi^2 c}{3L^2}\left[|A|^2 + |B|^2 - q^2\right]}$$

Notice that the antichiral part share the same $\{v_j^0, v_j^+, v_j^-\}$ with the chiral part, so their Möbius trans coffecients are same with the chiral part. The intergal is solved in Appendix 1.

So the problem left is to find the evolution of A and B. Again, to show how this works, we take the simplest setup as an example: Suppose the quench process is a periodic driving. which means in a given period $U(T) = \prod_{i=n}^{1} U_i$, and we repeat this time evolution afterwards: $U(mT) = U(T)^m$. Denote $\Pi = M_1 M_2 M_3 \cdots M_n = \begin{pmatrix} A_1 & B_1 \\ B_1^* & A_1^* \end{pmatrix}$, and $\Pi^m = \begin{pmatrix} A_m & B_m \\ B_m^* & A_m^* \end{pmatrix}$. To derive Π^n we only have to diagnolize Π:

$$\begin{vmatrix} \lambda - A_1 & -B_1 \\ -B_1^* & \lambda - A_1^* \end{vmatrix} = (\lambda - A_1)(\lambda - A_1^*) - |B_1|^2 = \lambda^2 - (A_1 + A_1^*)\lambda + 1 = 0$$

$$\Rightarrow \lambda = \frac{A_1 + A_1^* \pm \sqrt{(A_1 + A_1^*)^2 - 4}}{2} = \lambda_{1,2}$$

Notice that $A_1 + A_1^* = Tr(\Pi)$, so that:

① If $Tr(\Pi) > 2$:

In this case $\lambda_{1,2}$ are both real numbers and $\lambda_{1,2} > 1$. This leads to:

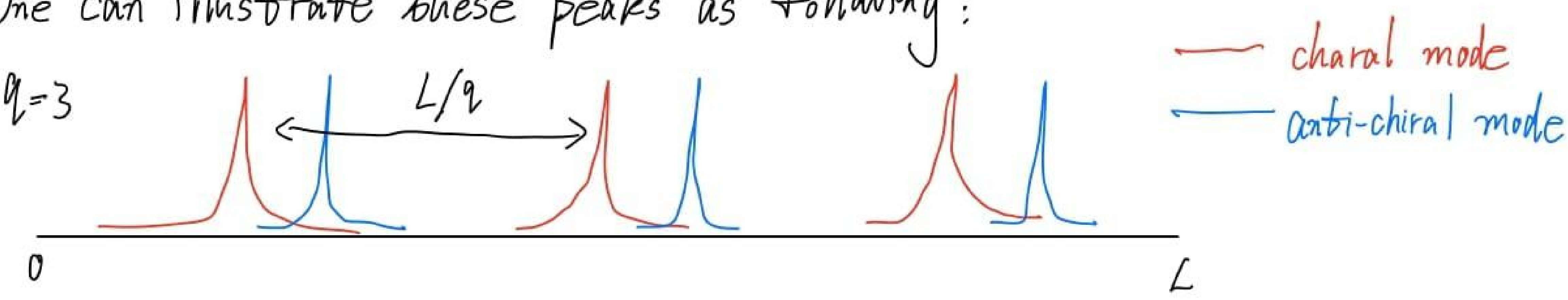
$$\Pi^m = \left[g \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} g^{-1} \right]^m = g \begin{pmatrix} \lambda_1^m & \\ & \lambda_2^m \end{pmatrix} g^{-1}$$

where g is the matrix that diagnolize Π. So $A_m, B_m \sim \lambda^m$, their value increase expotentially with time. So $E_m \sim |A_m|^2 + |B_m|^2$ also increases expotentially, which indicate a heating state.

Notice that $|A_m|^2 - |B_m|^2 = 1$. As A_m, B_m grows expotentially, when m is sufficiently large we can assume $|A_m|, |B_m| \gg 1$, then $|A_m| \approx |B_m|$. This will lead to some result to $T(z)$:

$$T(x, t = mT) = \frac{\pi^2 c}{6L^2} \left[\frac{q^2 - 1}{|A_m e^{-i\frac{2\pi q}{L}x} + B_m|^4} - q^2 \right]$$

Since $|A_m| \approx |B_m|$, $A_m e^{-i\frac{2\pi q x}{L}} + B_m = 0$ have solutions (Notice that in usual case this equation does not have solutions because $|A_m| > |B_m|$). Assume $\frac{B_m}{A_m} = -e^{-i\Delta\phi_m}$, then $e^{-i\frac{2\pi q x}{L}} = e^{-i\Delta\phi_m} \Rightarrow x = L\left(\frac{\Delta\phi_m}{2\pi q} - \frac{r}{q}\right)$, where r is an integer and $x \in [0, L]$. This indicates that every interval of $\frac{L}{q}$ has an energy density peak. We also have peaks for $\overline{T}(\bar{z})$, but these peaks are located at: $e^{i\frac{2\pi q x}{L}} = e^{-i\Delta\phi_m} \Longrightarrow$ $x = L\left(\frac{r}{q} - \frac{\Delta\phi_m}{2\pi q}\right)$, which are different from the position of peaks for charal mode. One can illustrate these peaks as following:



② If $Tr(\Pi) < 2$:

In this case $\lambda_{1,2}$ are both complex:

$$\lambda_{1,2} = \frac{A_1 + A_1^* \pm i\sqrt{4 - (A_1 + A_1^*)^2}}{2} \Longrightarrow |\lambda_{1,2}| = \left[(A_1 + A_1^*)^2 + 4 - (A_1 + A_1^*)^2\right]^{\frac{1}{2}} / 2 = 1$$

So $\lambda_{1,2}$ can be written as $e^{\pm i\gamma}$, and $\lambda_{1,2}^m = e^{\pm i m\gamma}$ oscillates with time. So the energy $E(mT)$ also oscillates with time, indicating a non-heating phase.

③ If $Tr(\Pi) = 2$:

In this case $\lambda_1 = \lambda_2 = 1$, $A_1 + A_1^* = 2$. Trivally we may think $\Pi = g\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} g^{-1} = I$, but this does not make sense since Π obviously can taken to be not I. For example, $A_1 = 2e^{i\frac{\pi}{3}}$, $A_1^* = 2e^{-i\frac{\pi}{3}}$, $B_1 = B_1^* = \sqrt{3}$, we still satisfy $\lambda_1 = \lambda_2 = 1$. The key point is that in this case Π can not be dignolized generally:

$$\begin{pmatrix} 2e^{i\frac{\pi}{3}} & \sqrt{3} \\ \sqrt{3} & 2e^{-i\frac{\pi}{3}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Longrightarrow (2e^{i\frac{\pi}{3}} - 1)x = -\sqrt{3}y \Longrightarrow \text{only one linear independent eigenvector}$$

(另一个方程与其等价)

In this situation Π can only be reduced to Jordan Block, for 2×2 case it would be like: $\Lambda = \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix}$, and still we have $\Pi = g\Lambda g^{-1}$.

$\Pi^m = g\Lambda^m g^{-1}$. The power of Λ can be given by:

$$\Lambda^2 = \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2a \\ & 1 \end{pmatrix}, \quad \Lambda^3 = \begin{pmatrix} 1 & 2a \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3a \\ & 1 \end{pmatrix}$$

$$\cdots\cdots \quad \Lambda^m = \begin{pmatrix} 1 & ma \\ & 1 \end{pmatrix}$$

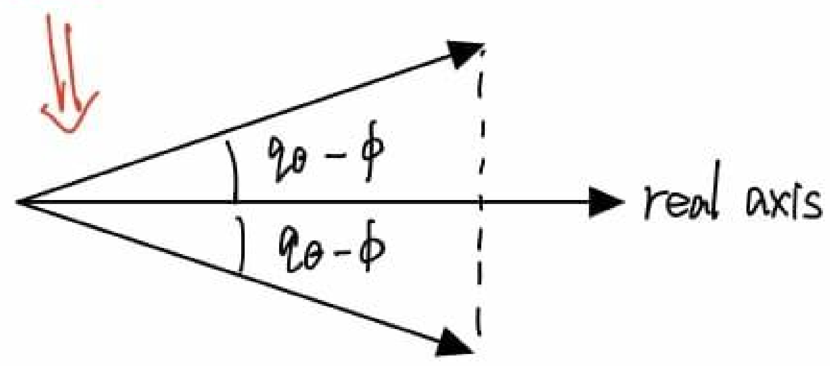
As we can see, now A_m, B_m would be propotional to m, so that $E(mT)$ linearly increases with time. This corresponds to the phase-transition state. We summarize here:

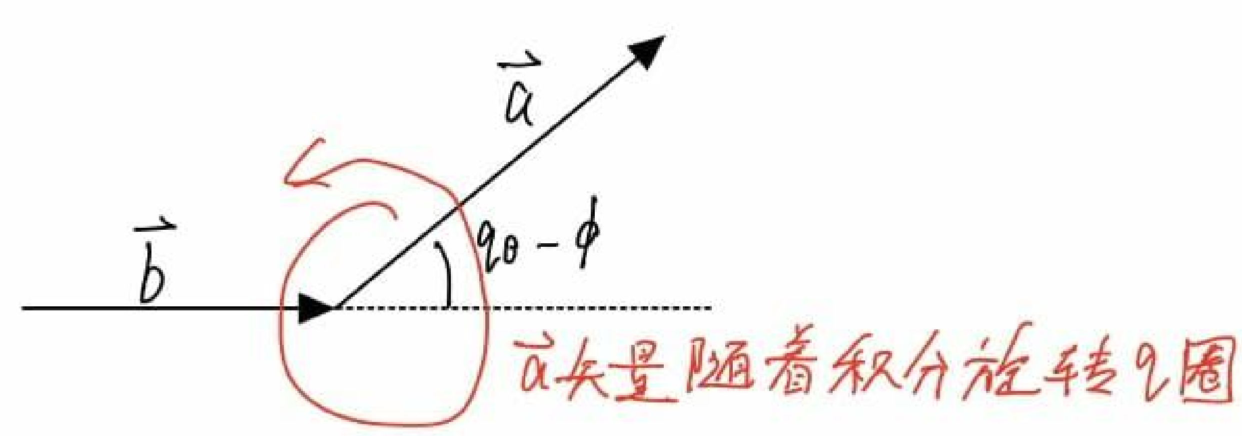
Phase	$Tr(\Pi)$	Energy growth	Property
Heating	>2	Expotential	Energy density peaks
Phase transition	$=2$	Linear	
Non-heating	<2	Oscillate	

Appendix:

1. $$\int_0^L \frac{1}{|Ae^{-i\frac{2\pi q}{L}x}+B|^4}\,dx = \int_0^L \frac{\frac{L}{2\pi}}{|Ae^{-i\frac{2\pi q}{L}x}+B|^4}\, d\left(\frac{2\pi x}{L}\right) = \frac{L}{2\pi}\int_0^{2\pi} \frac{d\theta}{|Ae^{-iq\theta}+B|^4}$$

Suppose $|A| = a$, $|B| = b$, $A = ae^{i(\phi+\varphi)}$, $B = be^{i\varphi}$, then $|Ae^{-iq\theta}+B| = |ae^{-i(q\theta-\phi)}+b|$
$= |ae^{i(q\theta-\phi)}+b|$





From the illustration, we can see the start point of the rotation does not affect the value of integral. We can prove it by:

$$\int_0^{2\pi} \frac{d\theta}{|ae^{i(q\theta-\phi)}+b|^4} = \frac{1}{q}\int_0^{2q\pi} \frac{d\theta}{|ae^{i(\theta-\phi)}+b|^4} = \frac{1}{q}\int_{-\phi}^{2q\pi-\phi} \frac{d\theta}{|ae^{i\theta}+b|^4}$$

$$= \frac{1}{q}\left(\int_0^{2q\pi-\phi} \frac{d\theta}{|ae^{i\theta}+b|^4} + \int_{-\phi}^{0} \cdots\right) = \frac{1}{q}\left(\int_0^{2q\pi-\phi} \cdots + \int_{2q\pi-\phi}^{2q\pi} \cdots\right) = \frac{1}{q}\int_0^{2q\pi} \cdots$$

$$= \int_0^{2\pi} \frac{d\theta}{|ae^{i\theta}+b|^4} = \int_0^{2\pi} \frac{d\theta}{(a^2+b^2+2ab\cos\theta)^2} = \oint \frac{\frac{dz}{iz}}{[a^2+b^2+ab(z+\frac{1}{z})]^2} = \frac{1}{i}\oint \frac{z\,dz}{[(a^2+b^2)z+ab(z^2+1)]^2}$$

$$= \frac{1}{i(ab)^2}\oint \frac{z\,dz}{[z^2+\frac{a^2+b^2}{ab}z+1]^2} = \frac{1}{i(ab)^2}\oint \frac{z\,dz}{(z-z_1)^2(z-z_2)^2}$$

Indeed z_1, z_2 can be solved: $z_{1,2} = \dfrac{-\frac{a^2+b^2}{ab} \pm \sqrt{(\frac{a^2+b^2}{ab})^2-4}}{2} = \dfrac{-(a^2+b^2)\pm 1}{2ab}$, where we use the relation $a^2-b^2=1$. By simple analysis we can find $z_2 = \dfrac{1-(a^2+b^2)}{2ab}$ is inside the unit circle. which means we need to find the residue at $z=z_2$.

$$\frac{z}{(z-z_1)^2(z-z_2)^2} = \frac{1}{(z-z_2)^2}(z_2+z-z_2)(z_2-z_1+z-z_2)^{-2}$$

$$= \frac{1}{(z-z_2)^2}(z_2+z-z_2)(z_2-z_1)^{-2}\left(1+\frac{z-z_2}{z_2-z_1}\right)^{-2} = \frac{1}{(z-z_2)^2}\left[z_2+(z-z_2)\right]\left[1-\frac{2(z-z_2)}{z_2-z_1}+\cdots\right]\frac{1}{(z_2-z_1)^2}$$

$$= \frac{1}{(z_2-z_1)^2}\left(\frac{z_2}{(z-z_2)^2} + \left(1-\frac{2z_2}{z_2-z_1}\right)\frac{1}{z-z_2} + \cdots\right)$$

So that: $\mathrm{Res}\left(\dfrac{z}{(z-z_1)^2(z-z_2)^2}, z_2\right) = \dfrac{1}{(z_2-z_1)^2}\cdot\dfrac{z_1+z_2}{z_2-z_1}\times(-1) = -\dfrac{z_1+z_2}{(z_2-z_1)^3} = \dfrac{(a^2+b^2)/ab}{(1/ab)^3}$

$= (a^2+b^2)\cdot(ab)^2$

$$\int_0^{2\pi} \frac{d\theta}{|ae^{i(q\theta-\phi)}+b|^4} = \frac{2\pi i}{i(ab)^2}(ab)^2\cdot(a^2+b^2) = 2\pi(a^2+b^2)$$

And finally: $\displaystyle\int_0^L \frac{1}{|Ae^{-i\frac{2\pi q}{L}x}+B|^4}dx = L(a^2+b^2) = L(|A|^2+|B|^2)$